

FOLIATIONS ON QUATERNION CR-SUBMANIFOLDS

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ABSTRACT. The purpose of this paper is to study the canonical foliations of a quaternion CR-submanifold of a quaternion Kähler manifold.

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1. INTRODUCTION

CR-submanifolds of Kähler manifolds were introduced in [4] by Bejancu. They appear as generalization both of totally real and of holomorphic submanifolds of Kähler manifolds.

This notion was further extended in the quaternion settings by Barros, Chen and Urbano ([3]). They consider CR-quaternion submanifolds of quaternion Kählerian manifolds as generalizations of both quaternion and totally real submanifolds.

If M is a quaternion CR-submanifold of a quaternion Kähler \overline{M} , then two distributions, denoted by D and D^\perp , are defined on M . It follows that D^\perp is always integrable (and of constant rank) i.e. is tangent to a foliation on M . Necessary and sufficient conditions are provided in order that this foliation became totally geodesic and Riemannian, respectively.

The paper is organized as follows: in Section 2 one reminds basic definitions and fundamental properties of quaternion CR-submanifolds of quaternion Kähler manifolds.

Section 3 allows techniques that will be proven to be useful to characterize the geometry of D and D^\perp ; conditions on total geodesicity are derived.

Section 4 deals with CR-quaternion submanifolds that are ruled by respect to D^\perp . Section 5 studies the case where D^\perp is tangent to a Riemannian foliation.

In the last section, QR-products in quaternion space forms are studied. A characterization of such submanifolds is given.

2. PRELIMINARIES ON QUATERNION CR-SUBMANIFOLDS

Let \overline{M} be a differentiable manifold of dimension n and assume that there is a rank 3-subbundle σ of $End(T\overline{M})$ such that a local basis $\{J_1, J_2, J_3\}$ exists of sections of σ satisfying:

$$(1) \quad \begin{cases} J_1^2 = J_2^2 = J_3^2 = -Id, \\ J_1 J_2 = -J_2 J_1 = J_3 \end{cases}$$

Then the bundle σ is called an almost quaternion structure on \overline{M} and $\{J_1, J_2, J_3\}$ is called canonical local basis of σ . Moreover, \overline{M} is said to be an almost quaternion manifold. It is easy to see that any almost quaternion manifold is of dimension $n = 4m$.

A Riemannian metric \bar{g} is said to be adapted to the almost quaternion structure σ if it satisfies:

$$(2) \quad \bar{g}(J_\alpha X, J_\alpha Y) = \bar{g}(X, Y), \forall \alpha \in \{1, 2, 3\},$$

for all vector fields X, Y on \bar{M} and any local basis $\{J_1, J_2, J_3\}$ of σ . Moreover, $(\bar{M}, \sigma, \bar{g})$ is said to be an almost quaternion hermitian manifold.

If the bundle σ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of \bar{g} , then $(\bar{M}, \sigma, \bar{g})$ is said to be a quaternion Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that:

$$(3) \quad \begin{cases} \bar{\nabla}_X J_1 = \omega_3(X)J_2 - \omega_2(X)J_3 \\ \bar{\nabla}_X J_2 = -\omega_3(X)J_1 + \omega_1(X)J_3 \\ \bar{\nabla}_X J_3 = \omega_2(X)J_1 - \omega_1(X)J_2 \end{cases}$$

for any vector field X on \bar{M} .

Let $(\bar{M}, \sigma, \bar{g})$ be an almost quaternion hermitian manifold. If $X \in T_x M, x \in M$, then the 4-plane $Q(X)$ spanned by $\{X, J_1 X, J_2 X, J_3 X\}$ is called a quaternion 4-plane. A 2-plane in $T_p M$ spanned by $\{X, Y\}$ is called half-quaternion if $Q(X) = Q(Y)$.

The sectional curvature for a half-quaternion 2-plane is called quaternion sectional curvature. A quaternion Kähler manifold is a quaternion space form if its quaternion sectional curvatures are equal to a constant, say c . It is well-known that a quaternion Kähler manifold $(\bar{M}, \sigma, \bar{g})$ is a quaternion space form (denoted $\bar{M}(c)$) if and only if its curvature tensor is:

$$(4) \quad \begin{aligned} \bar{R}(X, Y)Z &= \frac{c}{4} \{ \bar{g}(Z, Y)X - \bar{g}(X, Z)Y + \sum_{\alpha=1}^3 [\bar{g}(Z, J_\alpha Y)J_\alpha X \\ &\quad - \bar{g}(Z, J_\alpha X)J_\alpha Y + 2\bar{g}(X, J_\alpha Y)J_\alpha Z] \} \end{aligned}$$

for all vector fields X, Y, Z on \bar{M} and any local basis $\{J_1, J_2, J_3\}$ of σ .

Remark 2.1. For a submanifold M of a quaternion Kähler manifold $(\bar{M}, \sigma, \bar{g})$, we denote by g the metric tensor induced on M . If ∇ is the covariant differentiation induced on M , the Gauss and Weingarten formulas are given by:

$$(5) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y), \forall X, Y \in \Gamma(TM)$$

and

$$(6) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^\perp)$$

where B is the second fundamental form of M , ∇^\perp is the connection on the normal bundle and A_N is the shape operator of M with respect to N . The shape operator A_N is related to h by:

$$(7) \quad g(A_N X, Y) = \bar{g}(B(X, Y), N),$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

If we denote by \bar{R} and R the curvature tensor fields of $\bar{\nabla}$ and ∇ we have the Gauss equation:

$$(8) \quad \bar{g}(\bar{R}(X, Y)Z, U) = g(R(X, Y)Z, U) + \bar{g}(B(X, Z), B(Y, U)) - \bar{g}(B(Y, Z), B(X, U)),$$

for all $X, Y, Z, U \in \Gamma(TM)$.

A submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is called a quaternion CR-submanifold if there exists two orthogonal complementary distributions D and D^\perp on M such that:

i. D is invariant under quaternion structure, that is:

$$(9) \quad J_\alpha(D_x) \subseteq D_x, \forall x \in M, \forall \alpha = \overline{1, 3};$$

ii. D^\perp is totally real, that is:

$$(10) \quad J_\alpha(D_x^\perp) \subseteq T_x M^\perp, \forall \alpha = \overline{1, 3}, \forall x \in M.$$

A submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is a quaternion submanifold (respectively, a totally real submanifold) if $\dim D^\perp = 0$ (respectively, $\dim D = 0$).

We remark that condition ii. above implies that $J_\alpha(D_x^\perp)$ are in direct sum, for any local basis as in (1).

Definition 2.2. ([3]) Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$. Then M is called a QR-product if M is locally the Riemannian product of a quaternion submanifold and a totally real submanifold of \overline{M} .

Remark 2.3. If we denote by h^\perp and h the second fundamental forms of D^\perp and D , then we have the following two equations (see [5]):

i. D -Gauss equation:

$$(11) \quad \begin{aligned} g(R(X, Y)QZ, QU) &= g(R^D(X, Y)QZ, QU) + g(h(X, QZ), h(Y, QU)) \\ &\quad - g(h(Y, QZ), h(X, QU)), \end{aligned}$$

for all $X, Y, Z, U \in \Gamma(TM)$, where Q is the projection morphism of TM on D ;

ii. D^\perp -Gauss equation:

$$(12) \quad \begin{aligned} g(R(X, Y)Q^\perp Z, Q^\perp U) &= g(R^{D^\perp}(X, Y)Q^\perp Z, Q^\perp U) + g(h^\perp(X, Q^\perp Z), h^\perp(Y, Q^\perp U)) \\ &\quad - g(h^\perp(Y, Q^\perp Z), h^\perp(X, Q^\perp U)), \end{aligned}$$

for all $X, Y, Z, U \in \Gamma(TM)$, where Q^\perp is the projection morphism of TM on D^\perp .

Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$. The we say that:

i. M is D -geodesic if:

$$B(X, Y) = 0, \forall X, Y \in \Gamma(D).$$

ii. M is D^\perp -geodesic if:

$$B(X, Y) = 0, \forall X, Y \in \Gamma(D^\perp).$$

iii. M is mixed geodesic if:

$$B(X, Y) = 0, \forall X \in \Gamma(D), Y \in \Gamma(D^\perp).$$

We recall now the following result which we shall need in the sequel.

Theorem 2.4. ([3]) Let M be a CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$. Then:

i. The totally real distribution D^\perp is integrable.

ii. The quaternion distribution D is integrable if and only if M is D -geodesic.

A distribution D in a Riemannian manifold is called minimal if the trace of its second fundamental form vanishes.

We will illustrate here some of the techniques in this paper on the following proposition (see also [8], [11]).

Proposition 2.5. *If M is a CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$, then the quaternion distribution D is minimal.*

Proof. Take $X \in \Gamma(D)$ and $U \in \Gamma(D^\perp)$. Then we have:

$$\begin{aligned}
 g(\nabla_X X, U) &= \overline{g}(\overline{\nabla}_X X, U) \\
 &= \overline{g}(J_\alpha \overline{\nabla}_X X, J_\alpha U) \\
 &= \overline{g}(-(\overline{\nabla}_X J_\alpha)X + \overline{\nabla}_X J_\alpha X, J_\alpha U) \\
 &= \overline{g}(\omega_\beta(X)J_\gamma X - \omega_\gamma(X)J_\beta X + \overline{\nabla}_X J_\alpha X, J_\alpha U) \\
 &= \overline{g}(\overline{\nabla}_X J_\alpha X, J_\alpha U) \\
 (13) \quad &= \overline{g}(A_{J_\alpha U} J_\alpha X, X)
 \end{aligned}$$

and

$$g(\nabla_{J_\alpha X} J_\alpha X, U) = -\overline{g}(A_{J_\alpha U} X, J_\alpha X) = -\overline{g}(X, A_{J_\alpha U} J_\alpha X).$$

Now for the quaternion distribution D one takes local orthonormal frame in the form $\{e_i, J_1 e_i, J_2 e_i, J_3 e_i\}$ and summing up over i will give the assertion. \square

3. TOTALLY REAL FOLIATION ON A QUATERNION CR-SUBMANIFOLD

Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$. Then we have the orthogonal decomposition:

$$TM = D \oplus D^\perp.$$

We have also the following orthogonal decomposition:

$$TM^\perp = \mu \oplus \mu^\perp,$$

where μ is the subbundle of the normal bundle TM^\perp which is the orthogonal complement of:

$$\mu^\perp = J_1 D^\perp \oplus J_2 D^\perp \oplus J_3 D^\perp.$$

Since the totally real distribution D^\perp of a quaternion CR-submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ is always integrable we conclude that we have a foliation \mathfrak{F}^\perp on M with structural distribution D^\perp and transversal distribution D . We say that \mathfrak{F}^\perp is the canonical totally real foliation on M .

Theorem 3.1. *Let \mathfrak{F}^\perp be the canonical totally real foliation on a quaternion CR-submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$. The next assertions are equivalent:*

- i. \mathfrak{F}^\perp is totally geodesic;
- ii. $B(X, Y) \in \Gamma(\mu)$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$;
- iii. $A_N X \in \Gamma(D^\perp)$, $\forall X \in \Gamma(D^\perp)$, $N \in \Gamma(\mu^\perp)$;
- iv. $A_N Y \in \Gamma(D)$, $\forall Y \in \Gamma(D)$, $N \in \Gamma(\mu^\perp)$.

Proof. For $X, Z \in \Gamma(D^\perp)$ and $Y \in \Gamma(D)$ we have:

$$\begin{aligned}
\bar{g}(J_\alpha(\nabla_X Z), Y) &= -\bar{g}(\bar{\nabla}_X Z - B(X, Z), J_\alpha Y) \\
&= \bar{g}(-(\bar{\nabla}_X J_\alpha)Z + \bar{\nabla}_X J_\alpha Z, Y) \\
&= \bar{g}(\omega_\beta(X)J_\gamma Z - \omega_\gamma(X)J_\beta Z + \bar{\nabla}_X J_\alpha Z, Y) \\
&= \bar{g}(-A_{J_\alpha Z}X + \nabla_X^\perp J_\alpha Z, Y) \\
&= -g(A_{J_\alpha Z}X, Y)
\end{aligned}$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$, and taking into account of (7) we obtain:

$$(14) \quad \bar{g}(J_\alpha(\nabla_X Z), Y) = -\bar{g}(B(X, Y), J_\alpha Z).$$

i. \Rightarrow ii.

If \mathfrak{F}^\perp is totally geodesic, then $\nabla_X Z \in \Gamma(D^\perp)$, for $X, Z \in \Gamma(D^\perp)$ and from (14) we derive:

$$\bar{g}(B(X, Y), J_\alpha Z) = 0$$

and the implication is clear.

ii. \Rightarrow i.

If we suppose $B(X, Y) \in \Gamma(\mu)$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(D^\perp)$, then from (14) we derive:

$$\bar{g}(J_\alpha(\nabla_X Z), Y) = 0$$

and we conclude $\nabla_X Z \in \Gamma(D^\perp)$. Thus \mathfrak{F}^\perp is totally geodesic.

ii. \Leftrightarrow iii.

This equivalence is clear from (7).

iii. \Leftrightarrow iv.

This equivalence is true because A_N is a self-adjoint operator. \square

Corollary 3.2. *Let \mathfrak{F}^\perp be the canonical totally real foliation on a quaternion CR-submanifold M of a quaternion Kähler manifold $(\bar{M}, \sigma, \bar{g})$. If M is mixed geodesic, then \mathfrak{F}^\perp is totally geodesic.*

Proof. The assertion is clear from Theorem 3.1. \square

Corollary 3.3. *Let \mathfrak{F}^\perp be the canonical totally real foliation on a quaternion CR-submanifold M of a quaternion Kähler manifold $(\bar{M}, \sigma, \bar{g})$ with $\mu = 0$. Then M is mixed geodesic if and only if \mathfrak{F}^\perp is totally geodesic.*

Proof. The assertion is immediate from Theorem 3.1. \square

4. TOTALLY REAL RULED QUATERNION CR-SUBMANIFOLDS

A submanifold M of a Riemannian manifold (\bar{M}, \bar{g}) is said to be a ruled submanifold if it admits a foliation whose leaves are totally geodesic immersed in (\bar{M}, \bar{g}) .

Definition 4.1. A quaternion CR-submanifold of a quaternion Kähler manifold which is a ruled submanifold with respect to the foliation \mathfrak{F}^\perp is called totally real ruled quaternion CR-submanifold.

Theorem 4.2. *Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$. The next assertions are equivalent:*

- i. M is a totally real ruled quaternion CR-submanifold.*
- ii. M is D^\perp -geodesic and:*

$$B(X, Y) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp).$$

- iii. The subbundle μ^\perp is D^\perp -parallel, i.e.:*

$$\nabla_X^\perp J_\alpha Z \in \Gamma(\mu^\perp), \quad \forall X, Z \in D^\perp, \quad \alpha = \overline{1}, \overline{3}$$

and the second fundamental form satisfies:

$$B(X, Y) \in \Gamma(\mu), \quad \forall X \in \Gamma(D^\perp), \quad Y \in \Gamma(TM).$$

- iv. The shape operator satisfies:*

$$A_{J_\alpha Z} X = 0, \quad \forall X, Z \in D^\perp, \quad \alpha = \overline{1}, \overline{3}$$

and

$$A_N X \in \Gamma(D), \quad \forall X \in \Gamma(D^\perp), \quad N \in \Gamma(\mu).$$

Proof. i. \Leftrightarrow ii. For any $X, Z \in \Gamma(D^\perp)$ we have:

$$\begin{aligned} \overline{\nabla}_X Z &= \nabla_X Z + B(X, Z) \\ &= \nabla_X^{D^\perp} Z + h^\perp(X, Z) + B(X, Z) \end{aligned}$$

and thus we conclude that the leaves of D^\perp are totally geodesic immersed in \overline{M} iff $h^\perp = 0$ and M is D^\perp -geodesic.

The equivalence is now clear from Theorem 3.1.

- i. \Leftrightarrow iii. For any $X, Z \in \Gamma(D^\perp)$, and $U \in \Gamma(D)$ we have:

$$\begin{aligned} \overline{g}(\overline{\nabla}_X Z, U) &= \overline{g}(J_\alpha \overline{\nabla}_X Z, J_\alpha U) \\ &= \overline{g}(-(\overline{\nabla}_X J_\alpha)Z + \overline{\nabla}_X J_\alpha Z, J_\alpha U) \\ &= \overline{g}(\omega_\beta(X)J_\gamma Z - \omega_\gamma(X)J_\beta Z + \overline{\nabla}_X J_\alpha Z, J_\alpha U) \\ &= \overline{g}(-A_{J_\alpha Z} X + \nabla_X^\perp J_\alpha Z, J_\alpha U) \\ &= -g(A_{J_\alpha Z} X, J_\alpha U) \end{aligned}$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$, and taking into account of (7) we obtain:

$$(15) \quad \overline{g}(\overline{\nabla}_X Z, U) = -\overline{g}(B(X, J_\alpha U), J_\alpha Z).$$

On the other hand, for any $X, Z, W \in \Gamma(D^\perp)$ we have:

$$\begin{aligned} \overline{g}(\overline{\nabla}_X Z, J_\alpha W) &= \overline{g}(\overline{\nabla}_X Z + B(X, Z), J_\alpha W) \\ (16) \quad &= \overline{g}(B(X, Z), J_\alpha W). \end{aligned}$$

If $X, Z \in \Gamma(D^\perp)$ and $N \in \Gamma(\mu)$, then we have:

$$\begin{aligned} \overline{g}(\overline{\nabla}_X Z, N) &= \overline{g}(J_\alpha \overline{\nabla}_X Z, J_\alpha N) \\ &= \overline{g}(-(\overline{\nabla}_X J_\alpha)Z + \overline{\nabla}_X J_\alpha Z, J_\alpha N) \\ &= \overline{g}(\omega_\beta(X)J_\gamma Z + \omega_\gamma(X)J_\beta Z - J_\alpha \overline{\nabla}_X J_\alpha Z, N) \\ &= \overline{g}(\overline{\nabla}_X J_\alpha Z, J_\alpha N) \\ &= \overline{g}(-A_{J_\alpha Z} X + \nabla_X^\perp J_\alpha Z, J_\alpha N) \end{aligned}$$

and thus we obtain:

$$(17) \quad \bar{g}(\bar{\nabla}_X Z, N) = \bar{g}(\nabla_X^\perp J_\alpha Z, J_\alpha N).$$

Finally, M is a totally real ruled quaternion CR-submanifold iff $\bar{\nabla}_X Z \in \Gamma(D^\perp)$, $\forall X, Z \in \Gamma(D^\perp)$ and by using (15), (16) and (17) we deduce the equivalence.

ii. \Leftrightarrow iv.

This is clear from (7). \square

Corollary 4.3. *Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\bar{M}, \sigma, \bar{g})$. If M is totally geodesic, then M is a totally real ruled quaternion CR-submanifold.*

Proof. The assertion is clear from Theorem 4.2. \square

5. RIEMANNIAN FOLIATIONS AND QUATERNION CR-SUBMANIFOLDS

Let (M, g) be a Riemannian manifold and \mathfrak{F} a foliation on M . The metric g is said to be bundle-like for the foliation \mathfrak{F} if the induced metric on D^\perp is parallel with respect to the intrinsic connection on the transversal distribution D^\perp . This is true if and only if the Levi-Civita connection ∇ of (M, g) satisfies (see [5]):

$$(18) \quad g(\nabla_{Q^\perp Y} QX, Q^\perp Z) + g(\nabla_{Q^\perp Z} QX, Q^\perp Y) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

If for a given foliation \mathfrak{F} there exists a Riemannian metric g on M which is bundle-like for \mathfrak{F} , then we say that \mathfrak{F} is a Riemannian foliation on (M, g) .

Theorem 5.1. *Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\bar{M}, \sigma, \bar{g})$. The next assertions are equivalent:*

- i. *The induced metric g on M is bundle-like for totally real foliation \mathfrak{F}^\perp .*
- ii. *The second fundamental form B of M satisfies:*

$$B(U, J_\alpha V) + B(V, J_\alpha U) \in \Gamma(\mu) \oplus J_\beta(D^\perp) \oplus J_\gamma(D^\perp),$$

for any $U, V \in \Gamma(D)$ and $\alpha = 1, 2$ or 3 , where (α, β, γ) is an even permutation of $(1, 2, 3)$.

Proof. From (18) we deduce that g is bundle-like for totally real foliation \mathfrak{F}^\perp iff:

$$(19) \quad g(\nabla_U X, V) + g(\nabla_V X, U) = 0, \quad \forall X \in \Gamma(D^\perp), \quad U, V \in \Gamma(D).$$

On the other hand, for any $X \in \Gamma(D^\perp)$, $U, V \in \Gamma(D)$ we have:

$$\begin{aligned} g(\nabla_U X, V) + g(\nabla_V X, U) &= \bar{g}(\bar{\nabla}_U X - B(U, X), V) + \bar{g}(\bar{\nabla}_V X - B(V, X), U) \\ &= \bar{g}(\bar{\nabla}_U X, V) + \bar{g}(\bar{\nabla}_V X, U) \\ &= \bar{g}(-(\bar{\nabla}_U J_\alpha)X + \bar{\nabla}_U J_\alpha X, J_\alpha V) \\ &\quad + \bar{g}(-(\bar{\nabla}_V J_\alpha)X + \bar{\nabla}_V J_\alpha X, J_\alpha U) \\ &= \bar{g}(\omega_\beta(U)J_\gamma X - \omega_\gamma(U)J_\beta X + \bar{\nabla}_U J_\alpha X, J_\alpha V) \\ &\quad + \bar{g}(\omega_\beta(V)J_\gamma X - \omega_\gamma(V)J_\beta X + \bar{\nabla}_V J_\alpha X, J_\alpha U) \\ &= \bar{g}(\bar{\nabla}_U J_\alpha X, J_\alpha V) + \bar{g}(\bar{\nabla}_V J_\alpha X, J_\alpha U) \\ &= -g(A_{J_\alpha X} U, J_\alpha V) - g(A_{J_\alpha X} V, J_\alpha U) \end{aligned}$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$, and taking into account of (7) we derive:

$$(20) \quad g(\nabla_U X, V) + g(\nabla_V X, U) = -\bar{g}(B(U, J_\alpha V) + B(V, J_\alpha U), J_\alpha X),$$

for any $X \in \Gamma(D^\perp)$, $U, V \in \Gamma(D)$.

The proof is now complete from (19) and (20). \square

Corollary 5.2. *Let \mathfrak{F}^\perp be the canonical totally real foliation on a quaternion CR-submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ with $\mu = 0$. Then the induced metric g on M is bundle-like for \mathfrak{F}^\perp if and only if the second fundamental form B of M satisfies:*

$$B(U, J_\alpha V) + B(V, J_\alpha U) \in J_\beta(D^\perp) \oplus J_\gamma(D^\perp),$$

for any $U, V \in \Gamma(D)$ and $\alpha = 1, 2$ or 3 , where (α, β, γ) is an even permutation of $(1, 2, 3)$.

Proof. The assertion is immediate from Theorem 5.1. \square

Corollary 5.3. *Let \mathfrak{F}^\perp be the canonical totally real foliation on a quaternion CR-submanifold M of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ with $\mu = 0$. Then \mathfrak{F}^\perp is totally geodesic with bundle-like metric $g = \overline{g}|_M$ if and only if M is mixed geodesic and the second fundamental form B of M satisfies:*

$$B(U, J_\alpha V) + B(V, J_\alpha U) \in J_\beta(D^\perp) \oplus J_\gamma(D^\perp),$$

for any $U, V \in \Gamma(D)$ and $\alpha = 1, 2$ or 3 , where (α, β, γ) is an even permutation of $(1, 2, 3)$.

Proof. The proof follows from Corollary 3.3 and Corollary 5.2. \square

6. QR-PRODUCTS IN QUATERNION KÄHLER MANIFOLDS

From Theorem 2.4 we deduce that any D -geodesic CR-submanifold of a quaternion Kähler manifold admits a σ -invariant totally geodesic foliation, which we denote by \mathfrak{F} .

Proposition 6.1. *If M is a totally geodesic quaternion CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$, then M is a ruled submanifold with respect to both foliations \mathfrak{F} and \mathfrak{F}^\perp .*

Proof. The assertion follows from Corollary 4.3 and Theorem 2.4. \square

Theorem 6.2. *Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$. Then M is a QR-product if and only if the next three conditions are satisfied:*

- i. M is D -geodesic.
- ii. M is D^\perp -geodesic.
- iii. $B(X, Y) \in \Gamma(\mu)$, $\forall X \in \Gamma(D^\perp)$, $Y \in \Gamma(D)$.

Proof. The proof is immediate from Theorems 2.4 and 4.2. \square

Corollary 6.3. *Any totally geodesic quaternion CR-submanifold of a quaternion Kähler manifold is a QR-product.*

Proof. The assertion is clear. \square

Corollary 6.4. *Let M be a quaternion CR-submanifold of a quaternion Kähler manifold $(\overline{M}, \sigma, \overline{g})$ with $\mu = 0$. Then M is a QR-product if and only if M is totally geodesic.*

Proof. The assertion is immediate from Theorem 6.2. \square

Theorem 6.5. *Let M be a quaternion CR-submanifold of a quaternion space form $\overline{M}(c)$. If M is D -geodesic and ruled submanifold with respect to totally real foliation \mathfrak{F}^\perp , then M is a QR-product. Moreover, locally M is a Riemannian product $L \times L^\perp$, where L is a quaternion space form, having quaternion sectional curvature c , and L^\perp is a real space form, having sectional curvature $\frac{c}{4}$.*

Proof. Because \mathfrak{F}^\perp is a totally geodesic foliation, from (12) we obtain:

$$(21) \quad g(R(X, Y)Z, U) = g(R^{D^\perp}(X, Y)Z, U)$$

for any $X, Y \in \Gamma(TM)$, $Z, U \in \Gamma(D^\perp)$.

From Gauss equation and (21) we obtain:

$$(22) \quad \begin{aligned} \overline{g}(\overline{R}(X, Y)Z, U) &= g(R^{D^\perp}(X, Y)Z, U) + \overline{g}(B(X, Z), B(Y, U)) \\ &\quad - \overline{g}(B(Y, Z), B(X, U)). \end{aligned}$$

Since M is a ruled submanifold with respect to totally real foliation \mathfrak{F}^\perp , then M is D^\perp -geodesic (by Theorem 4.2) and from (22) we derive:

$$(23) \quad \overline{g}(\overline{R}(X, Y)Z, U) = g(R^{D^\perp}(X, Y)Z, U)$$

for any $X, Y, Z, U \in \Gamma(D^\perp)$.

Now, from (4) and (23) we derive:

$$(24) \quad g(R^{D^\perp}(X, Y)Y, X) = \overline{g}(\overline{R}(X, Y)Y, X) = \frac{c}{4},$$

for any orthogonal unit vector fields $X, Y \in \Gamma(D^\perp)$. Thus we conclude that the leaves of the totally real foliation \mathfrak{F}^\perp are of constant curvature $\frac{c}{4}$.

Since M is D -geodesic, from Gauss equation we obtain:

$$(25) \quad \overline{g}(\overline{R}(X', Y')Z', U') = g(R(X', Y')Z', U'),$$

for any $X', Y', Z', U' \in \Gamma(D)$.

On the other hand, if M is D -geodesic, then \mathfrak{F} is a totally geodesic foliation and from (11) we obtain:

$$(26) \quad g(R(X', Y')Z', U') = g(R^D(X', Y')Z', U'),$$

for any $X', Y' \in \Gamma(TM)$, $Z', U' \in \Gamma(D)$.

From (25) and (26) we deduce:

$$(27) \quad \overline{g}(\overline{R}(X', Y')Z', U') = g(R^D(X', Y')Z', U'),$$

for any $X', Y', Z', U' \in \Gamma(D)$.

Now, from (4) and (27) we derive:

$$(28) \quad g(R^{D^\perp}(X', J_\alpha X')J_\alpha X', X') = \overline{g}(\overline{R}(X', J_\alpha X')J_\alpha X', X') = c,$$

for any unit vector field $X' \in \Gamma(D)$. Thus we conclude that the leaves of the foliation \mathfrak{F} are of constant quaternion sectional curvature c .

The proof is now complete from (24) and (28). \square

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